Canonical & Standard Forms

CSIM601251

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Outline

- **Minterms and Maxterms**
- Relationship between Minterms and Maxterms: The Cannonical forms
- Conversion into Cannonical Sum-of-Minterm (SOM) or Product-of-Maxterm (POM) Representations
- Standard Form Sum-of-Products (SOP) and Product-of-Sum (POS)

Minterms and Maxterms (1/2)

- A minterm of n variables is a product term that contains n literals from all the variables.

  Example: On 2 variables x and y, the minterms are:

  x'·y', x'·y, x·y' and x·y

- A maxterm of n variables is a sum term that contains n literals from all the variables.

  Example: On 2 variables x and y, the maxterms are:

  x'+y', x'+y, x+y' and x+y

- In general, with n variables we have 2^n minterms and 2^n maxterms.

Minterms and Maxterms (2/2)

- The minterms and maxterms on 2 variables are denoted by m0 to m3 and M0 to M3 respectively.

x	y	Minterms		Maxterms	
		Term	Notation	Term	Notation
0	0	$x' \cdot y'$	m0	$x+y$	M0
0	1	$x' \cdot y$	m1	$x+y'$	M1
1	0	$x \cdot y'$	m2	$x'+y$	M2
1	1	$x \cdot y$	m3	$x'+y'$	M3

- Each minterm is the complement of the corresponding maxterm
  - Example: $m2 = x \cdot y'$

    $m2' = (x \cdot y')' = x' + (y')' = x' + y = M2$

Standard Order

- Minterms and maxterms are designated with a subscript
- The subscript is a number, corresponding to a binary pattern
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the same order (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms: $(a + b + \overline{c}), (a + b + c)$
  - Terms: $(b + a + c), a\overline{c}$b, and $(c + b + a)$ are NOT in standard order.
  - Minterms: $a\overline{b}c, abc, a\overline{b}\overline{c}$
  - Terms: $(a + c), \overline{b}c$, and $(\overline{a} + b)$ do not contain all variables

Purpose of the Index

- The <u>index</u> for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- For Minterms:
  - "1" means the variable is "Not Complemented" and
  - "0" means the variable is "Complemented".
- For Maxterms:
  - "0" means the variable is "Not Complemented" and
  - "1" means the variable is "Complemented".

Index Example in Three Variables

- Example: (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The Index 0 (base 10) = 000 (base 2) for three variables). All three variables are complemented for minterm 0 $(\overline{X}, \overline{Y}, \overline{Z})$ and no variables are complemented for Maxterm 0 (X, Y, Z).
  - Minterm 0, called m_0 is $\overline{X}.\overline{Y}.\overline{Z}$
  - Maxterm 0, called M_0 is $(X + Y + Z)$
  - Minterm 6 ?
  - Maxterm 6 ?

Index Examples – Four Variables

Index i	Binary Pattern	Minterm m_i	Maxterm M_i
0	0000	$\bar{a}\bar{b}\bar{c}\bar{d}$	$a+b+c+d$
1	0001	$\bar{a}\bar{b}\bar{c}d$	?
3	0011	?	$a+b+\bar{c}+\bar{d}$
5	0101	$\bar{a}b\bar{c}d$	$a+\bar{b}+c+\bar{d}$
7	0111	?	$a+\bar{b}+\bar{c}+\bar{d}$
10	1010	$a\bar{b}c\bar{d}$	$\bar{a}+b+\bar{c}+d$
13	1101	$ab\bar{c}d$	?
15	1111	$abcd$	$\bar{a}+\bar{b}+\bar{c}+\bar{d}$

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Minterm and Maxterm Relationship

- **Review: DeMorgan's Theorem**
  $\overline{x.y} = \overline{x} + \overline{y}$ **and** $\overline{x+y} = \overline{x}.\overline{y}$
- **Two-variable example:**
  $M_2 = \overline{x} + y$ **and** $m_2 = x.\overline{y}$
  Thus M_2 is the complement of m_2 and vice-versa.
- **Since DeMorgan's Theorem holds for *n* variables, the above holds for terms of *n* variables**
- **giving:**
  $M_i = \overline{m_i}$ **and** $m_i = \overline{M_i}$
  Thus M_i is the complement of m_i.

Function Tables for Both

- Minterms of 2 variables

x y	m_0	m_1	m_2	m_3
0 0	1	0	0	0
0 1	0	1	0	0
1 0	0	0	1	0
1 1	0	0	0	1

Maxterms of 2 variables

x y	M_0	M_1	M_2	M_3
0 0	0	1	1	1
0 1	1	0	1	1
1 0	1	1	0	1
1 1	1	1	1	0

- Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i.

Observations

- In the function tables:
  - Each minterm has one and only one 1 present in the 2^n terms (a minimum of 1s). All other entries are 0.
  - Each maxterm has one and only one 0 present in the 2^n terms All other entries are 1 (a maximum of 1s).
- We can implement any function by "**OR**ing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.
- We can implement any function by "**AND**ing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.
- This gives us two canonical forms:
  - **Sum of Minterms (SOM)**
  - **Product of Maxterms (POM)**

for stating any Boolean function.

Minterm Function Example

- Example: Find $F_1 = m_1 + m_4 + m_7$
- $F1 = \overline{x}\ \overline{y}\ z + x\ \overline{y}\ \overline{z} + x\ y\ z$

x y z	index	m1	+	m4	+	m7	= F1
0 0 0	0	0	+	0	+	0	= 0
0 0 1	1	1	+	0	+	0	= 1
0 1 0	2	0	+	0	+	0	= 0
0 1 1	3	0	+	0	+	0	= 0
1 0 0	4	0	+	1	+	0	= 1
1 0 1	5	0	+	0	+	0	= 0
1 1 0	6	0	+	0	+	0	= 0
1 1 1	7	0	+	0	+	1	= 1

Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$

Maxterm Function Example

- Example: Implement F1 in maxterms:

  $F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$

  $F_1 = (x + y + z) \cdot (x + \bar{y} + z) \cdot (x + \bar{y} + \bar{z}) \cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$ = F1
0 0 0	0	0 · 1 · 1 · 1 · 1 = 0
0 0 1	1	1 · 1 · 1 · 1 · 1 = 1
0 1 0	2	1 · 0 · 1 · 1 · 1 = 0
0 1 1	3	1 · 1 · 0 · 1 · 1 = 0
1 0 0	4	1 · 1 · 1 · 1 · 1 = 1
1 0 1	5	1 · 1 · 1 · 0 · 1 = 0
1 1 0	6	1 · 1 · 1 · 1 · 0 = 0
1 1 1	7	1 · 1 · 1 · 1 · 1 = 1

Maxterm Function Example

- $F(A,B,C,D) = M_3 \times M_8 \times M_{11} \times M_{14}$
- F(A, B,C,D) =

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Canonical Sum of Minterms

- Any Boolean function can be expressed as a Sum of Minterms.
  - For the function table, the minterms used are the terms corresponding to the 1's
  - For expressions, expand all terms first to explicitly list all minterms. Do this by "ANDing" any term missing a variable v with a term ($\mathbf{v} + \overline{\mathbf{v}}$).
- Example: Implement $\mathbf{f} = \mathbf{x} + \overline{\mathbf{x}}\ \overline{\mathbf{y}}$ as a sum of minterms.

  First expand terms: $\mathbf{f} = \mathbf{x}(\mathbf{y} + \overline{\mathbf{y}}) + \overline{\mathbf{x}}\ \overline{\mathbf{y}}$

  Then distribute terms: $\mathbf{f} = \mathbf{xy} + \mathbf{x}\overline{\mathbf{y}} + \overline{\mathbf{x}}\,\overline{\mathbf{y}}$

  Express as sum of minterms: $f = m_3 + m_2 + m_0$

Another SOM Example

- Example: $\mathbf{F = A + \overline{B}\,C}$
- There are three variables, A, B, and C which we take to be the standard order.
- Expanding the terms with missing variables:

- Collect terms (removing all but one of duplicate terms):
- Express as SOM:

Shorthand SOM Form

- From the previous example, we started with:

$$\mathbf{F = A + \overline{B}\,C}$$

- We ended up with:

$$F = m_1+m_4+m_5+m_6+m_7$$

- This can be denoted in the formal shorthand:

$$\mathbf{F(A, B, C) = \Sigma_m(1,4,5,6,7)}$$

- Note that we explicitly show the standard variables in order and drop the “m” designators.

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a Product of Maxterms (POM).
  - For the function table, the maxterms used are the terms corresponding to the 0's.
  - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law , "ORing" terms missing variable v with a term equal to and then applying the distributive law again. $v \cdot \bar{v}$
- Example: Convert to product of maxterms:

$$f(x, y, z) = x + \bar{x}\,\bar{y}$$

Apply the distributive law:

$$x + \bar{x}\,\bar{y} = (x + \bar{x})(x + \bar{y}) = 1 \cdot (x + \bar{y}) = x + \bar{y}$$

Add missing variable z:

$$x + \bar{y} + z \cdot \bar{z} = (x + \bar{y} + z)\,(x + \bar{y} + \bar{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- Convert to Product of Maxterms:

$$f(A,B,C) = A\overline{C} + BC + \overline{A}\,\overline{B}$$

- Use $x + yz = (x+y)\cdot(x+z)$ with $x = (A\overline{C} + BC)$, $y = \overline{A}$, and $z = \overline{B}$ to get:

$$f = (A\overline{C} + BC + \overline{A})(A\overline{C} + BC + \overline{B})$$

- Then use $x + \overline{x}y = x + y$ to get:

$$f = (\overline{C} + BC + \overline{A})(A\overline{C} + C + \overline{B})$$

and a second time to get:

$$f = (\overline{C} + B + \overline{A})(A + C + \overline{B})$$

- Rearrange to standard order,

$$f = (\overline{A} + B + \overline{C})(A + \overline{B} + C)$$ **to give** $f = M_5 \cdot M_2$

Another POM Example

Convert to Product of Maxterms: $F_1(A,B,C) = A\,C' + B\,C + A'B'$

A	B	C	F_1
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

$$F1 = M2 \bullet M5$$
$$= (A + B' + C) \bullet (A' + B + C')$$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given

$$F(x, y, z) = \sum_m(1,3,5,7)$$

$$\overline{F}(x, y, z) = \sum_m(0,2,4,6)$$

$$\overline{F}(x, y, z) = \prod_M(1,3,5,7)$$

Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
  - Find the function complement by swapping terms in the list with terms not in the list.
  - Change from products to sums, or vice versa.
- Example:Given F as before: $\mathbf{F(x,y,z) = \Sigma_m(1,3,5,7)}$
- Form the Complement: $\mathbf{\overline{F}(x,y,z) = \sum_m(0,2,4,6)}$
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $\mathbf{F(x,y,z) = \Pi_M(0,2,4,6)}$

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Standard Forms (1/2)

- Certain types of Boolean expressions lead to circuits that are desirable from implementation viewpoint.
- Two standard forms:
  - Sum-of-Products
  - Product-of-Sums
- Literals
  - A Boolean variable on its own or in its complemented form
  - Examples: x, x', y, y'
- Product term
  - A single literal or a logical product (AND) of several literals
  - Examples: x, x·y·z', A'·B, A·B, d·g'·v·w

Standard Forms (2/2)

- Sum term
  - A single literal or a logical sum (OR) of several literals
  - Examples: x, $x+y+z'$, $A'+B$, $A+B$, $c+d+h'+j$
- Sum-of-Products (SOP) expression
  - A product term or a logical sum (OR) of several product terms
  - Examples: x, $x + y \cdot z'$, $x \cdot y' + x' \cdot y \cdot z$, $A \cdot B + A' \cdot B'$, $A + B' \cdot C + A \cdot C' + C \cdot D$
- Product-of-Sums (POS) expression
  - A sum term or a logical product (AND) of several sum terms
  - Examples: x, $x \cdot (y+z')$, $(x+y') \cdot (x'+y+z)$, $(A+B) \cdot (A'+B')$, $(A+B+C) \cdot D' \cdot (B'+D+E')$
- Every Boolean expression can be expressed in SOP or POS.

Do it yourself

- Put the right ticks in the following table.

Expression	*SOP?*	*POS?*
X'·Y + X·Y' + X·Y·Z		
(X+Y')·(X'+Y)·(X'+Z')		
X' + Y + Z		
X·(W' + Y·Z)		
X·Y·Z'		
W·X'·Y + V·(X·Z + W')		

Standard Sum-of-Products (SOP)

- A sum of minterms form for *n* variables can be written down directly from a truth table.
  - Implementation of this form is a two-level network of gates such that:
    - The first level consists of *n*-input AND gates, and
    - The second level is a single OR gate (with fewer than 2^n inputs).
- This form often can be simplified so that the corresponding circuit is simpler.

Standard Sum-of-Products (SOP)

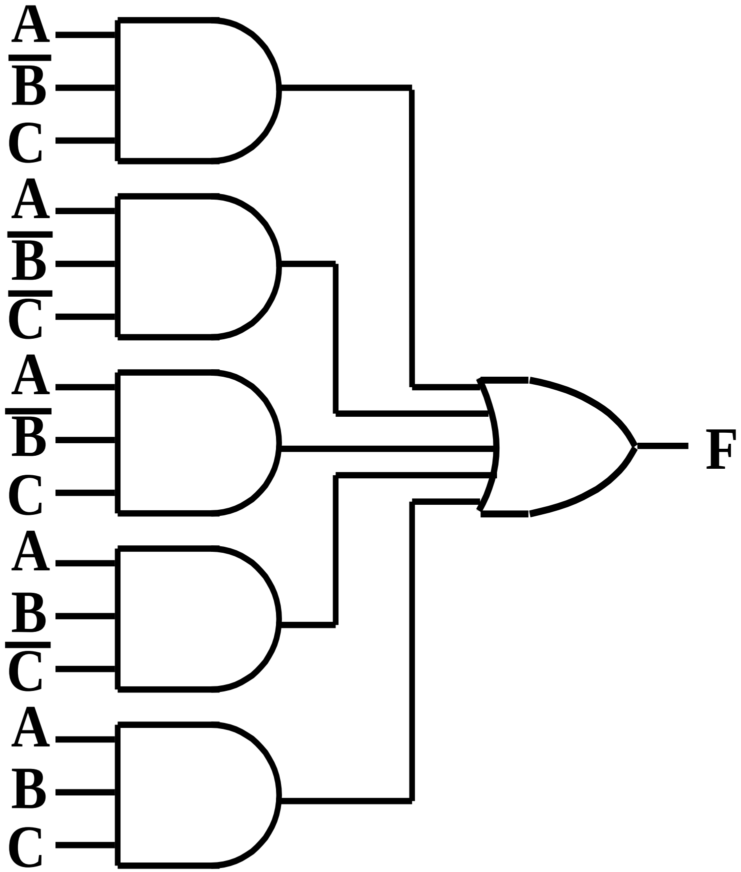
- A Simplification Example:
- $F(A, B, C) = \Sigma m(1,4,5,6,7)$
- Writing the minterm expression:

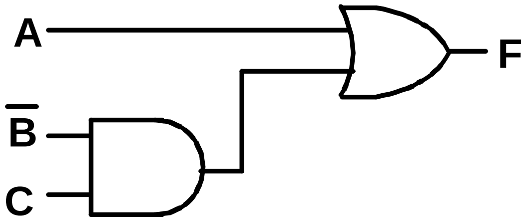
  $F = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,\overline{C} + A\,\overline{B}\,C + AB\overline{C} + ABC$
- Simplifying:

  F =
- Simplified F contains 3 literals compared to 15 in minterm F

AND/OR Two-level Implementation of SOP Expression

- The two implementations for F are shown below – it is quite apparent which is simpler!





SOP and POS Observations

- The previous examples show that:
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms.
  - Simpler equations lead to simpler two-level implementations
- Questions:
  - How can we attain a “simplest” expression?
  - Is there only one minimum cost circuit?
  - The next part will deal with these issues.